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COORDINATIZATION IN SUPERSTABLE THEORIES. I. STATIONARY TYPES

BY

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ABSTRACT. Suppose T is superstable and P is a complete type over some finite set with $U(p) = \alpha + 1$ for some α. We show how to associate with p an incidence geometry which measures the complexity of the family of extensions of p of rank α. When p is stationary we give a characterization of the possible incidence geometries. As an application we prove

THEOREM. *Suppose M is superstable and has only one 1-type $p \in S(\varnothing)$. Further suppose p is stationary with $U(p) = \alpha + 1$ for some α. Then one of the following holds:*

(i) *There is an equivalence relation $E \subset M^2$ with infinitely many infinite classes definable over $\varnothing$.*

(ii) *M is the algebraic closure of a set of Morley rank 1. In particular, M is $\aleph_0$-stable of finite rank.*

1. Preliminaries. We assume a basic knowledge of stability theory as found in [**M**, **P**, or **Sh**]. For the most part our notation follows [**M**]. In particular, we use $A \downarrow_C B$ to mean $t(A/B \cup C)$ does not fork over C. If p is stationary and there is a type $q \in S(B)$ such that p is parallel to q, we write $q = p|B$. For $t(a/A) = t(b/A)$ we write $a \equiv b(A)$; $a \overset{s}{\equiv} b(A)$ denotes $\operatorname{stp}(a/A) = \operatorname{stp}(b/A)$. We say $p \in S(A)$ *has finite multiplicity* if $\{\operatorname{stp}(a/A): a \text{ realizes } p\}$ contains finitely many inequivalent types.

We assume every set is a subset of a large saturated model $\mathfrak{C}$ called the monster model. Every model is considered to be an elementary submodel of $\mathfrak{C}$. See [**Sh**, p. 7] for a more complete discussion. In this paper we work in $\mathfrak{C}^{eq}$, where there are names for the classes of definable equivalence relations. A detailed discussion is found in [**Sh**, III, §6].

The remaining terminology is as follows. For p a type (not necessarily complete) and A a set, we let $p(A) = \{b \in A : b \text{ realizes } p\}$. We say $H \subset M$ is *A-definable* if there is a formula φ over A such that $H = \varphi(M)$. H is *0-definable* if it is definable over $\varnothing$. If there is a set $A \subset M$ and $p \in S_1(A)$ such that $H = p(M)$, then H is said to be *transitive*. Sets A and B are *conjugate over C in M* if $A, B, C \subset M$ and there is an automorphism α of M such that α pointwise fixes C and $\alpha(A) = B$. With the same notation, if p is a type over A, q a type over B, and $\alpha(p) = q$, we say that p and q are *conjugate over C*.

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We will occasionally attribute to definable sets properties normally reserved for formulas; similarly, if $H = p(M)$ for some complete p. For example, the rank of H is the rank of p.

We will want to think of definable subsets of a model as being actual elements of the model. With imaginary elements this can be done. Say $H = \varphi(M)$ for $\varphi(x, \bar{a})$ a formula over M. Let $E(\bar{y}, \bar{y}')$ be the equivalence relation $\forall x(\varphi(x, \bar{y}) \leftrightarrow \varphi(x, \bar{y}'))$. In M^{eq} there are names for the equivalence classes of E; $\mathscr{A}$ is the set of such classes. We define an incidence relation $\varepsilon \subset M \times \mathscr{A}$ by $b \,\varepsilon\, h$ iff there is a $\bar{c}$ such that $h = \bar{c}/E$ and $\varphi(b, \bar{c})$ holds. Of course, ε is definable in M^{eq}. Then if $h = \bar{a}/E$ we could think of the formula $x \varepsilon h$ in the same way we think of $x \in H$. We call h a *name* for H.

LEMMA 1.1. *Suppose h names $\varphi(M, \bar{a})$, and $t(\bar{a}/\varnothing)$ is stationary. Then $t(h/\varnothing)$ is stationary.*

PROOF. Notice that h is the only element satisfying $t(h/\bar{a})$. This implies $t(h\bar{a}/\varnothing)$ is stationary. Using [**Sh**, III, 4.15] we show that every nonforking extension of $t(h/\varnothing)$ gives rise to a nonforking extension of $t(h\bar{a}/\varnothing)$. The lemma follows directly.

Now we discuss other preliminaries in greater depth.

1.1. *Incidence geometries.* Following [**D**] we define an *incidence geometry* to be a triple $(P, \mathscr{B}, \varepsilon)$, where $P, \mathscr{B}$ are sets and $\varepsilon \subset P \times \mathscr{B}$. The elements P and $\mathscr{B}$ are called *points* and *blocks*, respectively, and ε is called the *incidence relation*. For example, we may have P as the points and $\mathscr{B}$ as the hyperplanes in a projective geometry. Often we will be discussing geometries of the form $(P, Q; I)$, where P, Q are sets of realizations of complete types and $I = I_1 \upharpoonright P \times Q$ for some formula I_1. In this case we write $(P, Q; I_1)$, letting it be understood that the incidence relation is actually $I_1 \upharpoonright P \times Q$.

Incidence geometries first arose in stability theory in [**La**] where Lachlan reduced a conjecture to the nonexistence of $\aleph_0$-categorical pseudoplanes. For more on the role of pseudoplanes in stability theory see [**Z, CHL**].

For the remainder of the section we assume T is a fixed countable complete superstable theory.

1.2. *Rank.* In [**L**] Lascar introduced the rank U on complete types in a superstable theory. See also [**LP**; **M** §A]. The other most commonly used rank in superstable theories is what Shelah calls $R(-, L, \infty)$. We denote this rank simply by $R(-)$. We work mostly with U-rank because of the nice inequalities it obeys. However, U-rank cannot be applied to formulas. When we need to speak of the rank of a formula, we use R. We use the following identities on U-rank. See Theorems 5.6 and 5.8 of [**L**].

LEMMA 1.2. (i) *Suppose there is an $n \in \omega$ such that $U(b/A) = U(b/A \cup c) + n$. Then $U(c/A) = U(c/A \cup b) + n$.*

(ii) *If $U(b/A) < \omega$, then $U(bc/A) = U(c/A \cup b) + U(b/A)$.*

1.3. *Canonical representatives.* Given a $p \in S(\mathfrak{C})$ there are many $\varphi \in p$ with $R(p) = R(\varphi)$. We describe how to pick one which is particularly well behaved. The next lemma follows easily from [**Sh**, p. 89, Remark]. Av(− / −) denotes the average type of a set of indiscernibles. See [**Sh**, p. 88, or **P**, 7.34].

LEMMA 1.3. *Let* $p \in S(\mathfrak{C})$ *and let* I *be a set of indiscernibles such that* $\mathrm{Av}(I/\mathfrak{C}) = p$. *Then there is an* $I_0 \subset I$ *such that* p *does not fork over* I_0.

Since T is superstable, we may pick I_0 finite and of minimal cardinality such that p does not fork over I_0. Since I is indiscernible, any subset of the same size shares this property. Let $\bar{a}$ enumerate I_0. Then there is a formula $\varphi(x, \bar{a}) \in p$ such that $R(p) = R(\varphi)$. Since any b realizing p satisfies "$b\bar{a}$ is an indiscernible sequence", we may assume $\varphi(x, \bar{y})$ is invariant under permutation of the variables. That is, if $\bar{a} = a_0\bar{a}'$, then $t(a_0/b\bar{a}')$ contains $\varphi(x, b\bar{a}')$.

We call such a $\varphi(x, \bar{a})$ a *canonical representative* of p. If p_0 is such that $p = p_0|\mathfrak{C}$, then we also say that φ is a canonical representative of p_0. We call $|\bar{a}|$ the *complexity* of φ (or p).

1.4. *Normalization*. The technique of normalizing a formula was developed by Lachlan in [**La**]. Most normalization theorems involve a rank S and have the following form. Given $\varphi(x, a)$ of rank α there is a $\varphi^*(x, a)$ such that

$$S(\neg(\varphi(x, a) \leftrightarrow \varphi^*(x, a))) < \alpha;$$

and for $b \equiv a$,

$$(*) \qquad \text{if} \quad S(\neg(\varphi(x, a) \leftrightarrow \varphi(x, b))) < \alpha, \quad \text{then} \quad \varphi^*(x, a) \leftrightarrow \varphi^*(x, b).$$

Harnik and Harrington have formulated a general normalization theorem with an arbitrary equivalence relation replacing the one defined by (∗).See [**HH**, Chapter 9]. We find, however, that the formulation in [**HH**] is difficult to apply in the context of this paper. The difficulty lies mostly in that U-rank is not continuous (i.e., is defined only for complete types), so we cannot define an equivalence relation using (∗). We offer instead a different normalization theorem. The following is extracted from [**B**], where 1.4 is proved.

Basic assumptions. Fix a formula $\varphi(x, a)$. Assume that associated with $\varphi(x, a)$ we have a set $P_a \subset S(\mathfrak{C})$ satisfying the following:

(N1) $\varphi(x, a) \in q$ for all $q \in P_a$.

(N2) P_a is closed under a-automorphisms.

(N3) $|P_a| < \lambda$ for some $\lambda < |M|$.

For each $b \equiv a$ let P_b be the set of images of P_a under some automorphism taking a to b. We further require

(N4) $P_a = P_b$ iff, for all $q \in P_a$, $\varphi(x, b) \in q$ and, for all $q \in P_b$, $\varphi(x, a) \in q$.

Let Ψ be the set of formulas equivalent to a positive Boolean combination of conjugates of $\varphi(x, a)$. We can extend the notion of a distinguished extension to elements of Ψ in a natural way by the following recursion:

(A) If $b \equiv a$, $Q(\varphi(x, b)) = P_b$.

(B) If $\theta, \tau \in \Psi$ and $Q(\theta)$ and $Q(\tau)$ have been defined, $Q(\theta \wedge \tau) = Q(\theta) \cap Q(\tau)$, and $Q(\theta \vee \tau) = Q(\theta) \cup Q(\tau)$.

Let $\mathscr{P} = \{Q(\theta) : \theta \in \Psi\}$.

DEFINITION. We say $\theta(x, c) \in \Psi$ is *normal with respect to* $\mathscr{P}$ if, for all $d \equiv c$ such that $Q(\theta(x, c)) = Q(\theta(x, d)), \theta(M, c) = \theta(M, d)$.

THEOREM 1.4 (NORMALIZATION THEOREM). *Suppose* $\varphi(x, a)$ *and* P_a *satisfy* (N1)–(N4). *Then there is a formula* $\varphi^*(x, a)$ *such that*

(i) $\varphi^*(x, a) \in \Psi$,

(ii) $P_a = Q(\varphi^*(x, a))$,

(iii) $\varphi^*(x, a)$ *is normal with respect to* $\mathscr{P}$.

We call $\varphi^*(x, a)$ a *normalization* of $\varphi(x, a)$ with respect to $\mathscr{P}$.

An important consequence of Theorem 1.4 is the following. Define an equivalence relation on $\mathfrak{C}$ by

$$E(x, y) \leftrightarrow \forall z(\varphi^*(z, x) \leftrightarrow \varphi^*(z, y)).$$

Then if b, c realize $t(a)$, $E(b, c) \leftrightarrow P_b = P_c$.

1.5. *Definable families.* Central to the notion of a coordination is the notion of a family of conjugate sets. For $\varphi(x, \bar{a})$ any formula, we define the *family of conjugates of* φ to be

$$\mathfrak{F} = \{\varphi(\mathfrak{C}, \bar{b}) : \bar{b} \equiv \bar{a}\}.$$

We call $t(\bar{a})$ the *index type* of $\mathfrak{F}$. Notice that $\mathfrak{F}$ is the orbit of $\varphi(\mathfrak{C}, \bar{a})$ under the automorphisms of $\mathfrak{C}$.

Working in $\mathfrak{C}^{\mathrm{eq}}$ we will have names for the elements of $\mathfrak{F}$. We denote $\varphi(\mathfrak{C}, \bar{b})$ by $A_{\bar{b}}$, $\mathscr{C}_{\bar{b}}$, $\mathscr{G}_{\bar{b}}$, etc., when φ is clear from context. Notice that there may be some redundancy in the indexing—$A_{\bar{a}}$ may equal $A_{\bar{b}}$. When this is the case we can define an equivalence relation by $\bar{a} \approx \bar{b}$ iff $A_{\bar{a}} = A_{\bar{b}}$. It is then easy to find another formula $\varphi'(x, y)$ such that $\varphi(\mathfrak{C}, \bar{a}) = \varphi'(\mathfrak{C}, \bar{a}/\approx)$. Then we can index $\mathfrak{F}$ by $t(\bar{a}/\approx)$ instead of $t(\bar{a})$. *We will always assume such a reduction has been done.* This is important in results involving the rank of the index type.

Let $\mathfrak{F}$ be the family of conjugates of φ, and let φ^* be the normalization of φ with respect to some fixed $\mathscr{P}$. Then $\mathfrak{F}^*$ denotes the family of conjugates of φ^* and is called the *normalization* of $\mathfrak{F}$.

1.6. *Coordinatization.* In **[CHL]** one of the most important technical devices is what is called a coordinatization. Its definition is as follows. Let M be $\aleph_0$-categorical and $\aleph_0$-stable. If P, A are two infinite 0-definable subsets of M, we say that A *coordinatizes* P if

1. A is an atom, and
2. For all x in P, $\mathrm{acl}(x) \cap A \neq \varnothing$.

PROPOSITION 1.5 (COORDINATIZATION THEOREM [**CHL**, THEOREM 4.1]). *Let M be $\aleph_0$-categorical, $\aleph_0$-stable, and transitive. Then in M^{eq} there is a* 0*-definable set A of Morley rank* 1 *which coordinatizes M.*

We must change the definition of a coordinatization if it is to be applicable to an arbitrary superstable theory. To motivate the definition given in the next section, we describe how the set A arises in Proposition 1.5.

Let M be as in Proposition 1.5. Let rk(–) denote Morley rank. It is known that $\mathrm{rk}(M) < \omega$, say $\mathrm{rk}(M) = n + 1$ for $n > 0$. Suppose $\varphi(x, \bar{a})$ has rank n and degree 1.

Let $\mathfrak{F}$ be the family of conjugates of φ, and let $\mathfrak{F}^*$ be its normalization. Then by 3.3 of **[CHL]** the rank of the index set of $\mathfrak{F}^*$ is 1. Let A be the set of names for elements of $\mathfrak{F}^*$. Then A is a 0-definable subset of M^{eq} of rank 1. For $x \in M$, $\text{acl}(x) \cap A$ contains the names $A_{\bar{b}}$ such that $x \in \varphi^*(M, \bar{b})$.

This description shows more clearly why Proposition 1.5 is called the Coordinatization Theorem. A point in M is located by specifying which conjugates of φ it lies in. This is a rather local coordinatization in that we look only at the conjugate of φ. But presumably this process can be iterated through formulas of smaller rank to obtain a more complete "dissection" of the structure.

2. Formulation of coordinatization. This section contains only the definition of the incidence geometry alluded to in the abstract. The main theorem is stated prior to the examples at the end of the section.

Perhaps the best motivation for the forthcoming material is an historical description of how it arose. One of the most important results in **[CHL]** is

PROPOSITION 2.1. *Let M be $\aleph_0$-categorical and $\aleph_0$-stable, and let $p \in S_1(M)$. Then there is a single element $a \in M$ such that p does not fork over a.*

The proof of Proposition 2.1 depends heavily on the Cherlin-Zil′ber classification theorem for $\aleph_0$-categorical strongly minimal sets **[CHL**, 2.1]. It was our hope to prove Proposition 2.1 more directly. If we argue towards a contradiction, taking a failure of highest rank, we quickly specialize to a type $p \in S_1(M)$ of rank n, where M is transitive of rank $n + 1$. Let $\varphi(x, \bar{a}) \in p$ be a canonical representative. By assumption, $|\bar{a}| > 1$, and observe that since M has rank $n + 1$ and φ has rank n we may take $\bar{a}$ to be independent. We discovered that such a formula gives rise to a complicated family of conjugates $\mathfrak{F}$ in M. For example, in M^{eq} there is a definable pseudoplane obtained in a natural way using $\mathfrak{F}$.

This investigation led to the intuition that there is a direct relationship between the number of parameters in $\varphi(x, \bar{a})$ and the geometrical complexity of the family of conjugates of φ. We also observed that this correspondence could be formulated for an arbitrary superstable theory and that, when M is transitive, the set of points in the resulting incidence geometry is $\text{acl}(A)$ for some A of Morley rank 1. In retrospect we have come to view the introduction of the family of conjugates as a coordinatization of M with respect to φ. It is a coordinatization in the sense that an element of M is located by the conjugates it lies in. The incidence geometry $(M, \mathfrak{F}, \varepsilon)$ gives the structure on the set of coordinates.

For the remainder of this paper let T be a countable superstable theory and M an uncountable saturated model of T. First we must formulate what we mean by a coordinatization in this context. Let p_0 be a complete 1-type with $U(p_0) = \alpha + 1$ for some $\alpha > 0$. By adding constants to the language we may assume $\text{dom}(p_0) = \varnothing$. Let

(1) $H = p_0(M)$ and $p \in S(M)$ be an extension of p_0 of U-rank α; $\sigma(x, \bar{a})$ a canonical representative of p; and $p_1 = p|\bar{a}$.

Later we will place more requirements on σ. Remember we chose $\bar{a}$ to have minimal length under the above restrictions. Observe that $\bar{a}$ is independent: if $\bar{a} = b\bar{a}_1$ is dependent, then $U(b/\bar{a}_1) < \alpha + 1$; since $U(p) = \alpha$ we have $U(p) = U(p|\bar{a}_1)$, contradicting the minimality of $\bar{a}$. Let $\mathfrak{F}$ be the family of conjugates of $\sigma(x, \bar{a})$ in M. Finally, let $\mathfrak{H} = \langle H, \mathfrak{F}, \varepsilon \rangle$, formalized in M^{eq} as described in §1. We call $\mathfrak{H}$ the *coordinate pregeometry of H with respect to* σ (or p). We call $k = |\bar{a}|$ the *complexity* of $\mathfrak{H}$. Notice that we are not considering $\mathfrak{H}$ as carrying the structure induced by M. $\mathfrak{H}$ consists solely of the two sorts H and $\mathfrak{F}$ and the relation $\varepsilon \subset H \times \mathfrak{F}$; i.e., $\mathfrak{H}$ is an incidence geometry.

For $a \in H$ we call $\{\ell \in \mathfrak{F}: a \,\varepsilon\, \ell\}$ the *coordinates of* a, denoted crd(a).

From here on we denote the elements of $\mathfrak{F}$ by ℓ, ℓ^0, $\mathscr{g}$, etc.; $\ell_{\bar{b}}$ denotes the name for the set $\sigma(M, \bar{b})$.

For $\ell \in \mathfrak{F}$ we will mostly be interested in those $a \,\varepsilon\, \ell$ which have maximal rank over ℓ. With notation as in (1) let b realize p_1, $\ell = \ell_{\bar{a}}$, and $\gamma = U(\ell/b)$. Notice that the rank identity Lemma 1.2(i) implies $U(\ell/\varnothing) = \gamma + 1$. We define the following for $\ell \in \mathfrak{F}$ and $a \in H$:

$$(2) \qquad \underset{\sim}{\ell} = \bigcup_{n<\omega} \{\bar{a} \in H^n: \bar{a} \text{ is } \ell\text{-independent}, a_i \,\varepsilon\, \ell, \text{ and } U(a_i/\ell) = \alpha \text{ for } i < n\}.$$

$$(3) \qquad \underset{\sim}{a} = \bigcup_{n<\omega} \{\bar{\ell} \in \mathfrak{F}^n: \bar{\ell} \text{ is } a\text{-independent}, a \,\varepsilon\, \ell_i, \text{ and } \gamma = U(\ell_i/a) \text{ for } i < n\}.$$

$$(4) \qquad R_a = \{q \in S(M^{\mathrm{eq}}): a \,\varepsilon\, x \text{ is in } q, U(q) = \gamma, \text{ and } q \supset t(\ell)\}.$$

Notice that $a \in \underset{\sim}{\ell}$ if and only if $\ell \in \underset{\sim}{a}$.

LEMMA 2.2. *We may assume the canonical representative* σ *was chosen so that* ε *satisfies the following: for all* $a, b \in H$,

$$R_a = R_b \quad \textit{iff} \quad M \models \forall y(a \,\varepsilon\, y \leftrightarrow b \,\varepsilon\, y).$$

This lemma is proved by normalization; the details are deferred to the next section. Let E be the equivalence relation defined in M by

$$(5) \qquad E(z, z') \quad \text{iff} \quad \forall y(z \,\varepsilon\, y \leftrightarrow z' \,\varepsilon\, y).$$

On H, $E(a, b)$ is interpreted as saying "a and b have the same coordinates".

There is a natural way to induce an incidence geometry on H/E. Define ε_0 in M by

$$(6) \qquad (x/E)\, \varepsilon_0\, y \quad \text{iff} \quad x \,\varepsilon\, y.$$

Then $\mathfrak{H}/E = \langle H/E, \mathfrak{F}, \varepsilon_0 \rangle$. We will usually denote H/E by P and $\mathfrak{H}/E$ by $\mathfrak{G}$. We call $\mathfrak{G}$ *the coordinate geometry of H with respect to* σ (or p).

Let $n \in \omega$ satisfy

(7) $n + 1$ is the maximal m such that for $a \in H$ there is a $\bar{\ell} \in \underset{\sim}{a}$ of length m with $\bar{\ell}$ independent over $\varnothing$.

THEOREM 1. *Suppose H is stationary. There is a set A definable in* $\mathfrak{G}$ *such that* $P = \mathrm{acl}(A)$, *where* acl( ) *is computed in the structure* $\mathfrak{G}$. *Furthermore*, $A = P \cap A_0$ *for some definable* A_0 *in* M^{eq}, *the U-rank of A in* M^{eq} *is* 1, *and A has finite multiplicity.*

This theorem can be greatly simplified if the only 1-type in M is p_0. See Theorem 2 in the next section. We close this section with a couple of examples of what $\mathfrak{G}$ may look like.

EXAMPLE 1. Let B be an $(n+1)$-dimensional vector space over an infinite algebraically closed field. Let P be the set of 1-dimensional subspaces, $\mathfrak{F}$ the set of n-dimensional subspaces, and $\mathfrak{G} = \langle P, \mathfrak{F}, \subset \rangle$. That is, $\mathfrak{G}$ is the geometry of the hyperplanes in n-dimensional projective geometry over the field.

To see that $\mathfrak{G}$ arises as the coordinate geometry of some H, simply take $M = \mathfrak{G}$, $H = P$, and σ as the obvious formula. The coordinatization process yields $\mathfrak{G}$ as the coordinate geometry. This example shows that n can be any finite number.

EXAMPLE 2. Let A be some uncountable set, and let B be the set of m-element subsets of A. Define $R \subset B \times B$ by $R(\{a_1,\dots,a_m\},\{b_1,\dots,b_m\})$ iff $a_i = b_j$ for some i, j. Let $M = \langle B, R\rangle$, and $\mathfrak{G} = \langle P, \mathfrak{F}, \varepsilon_0\rangle$, the coordinate geometry of M with respect to R. Then $P = B$ has rank m, and the complexity k is 1. This $\mathfrak{G}$ is easier to picture if, for $\ell \in \mathfrak{F}$, we add names for the m degree-1 components of $\underset{\sim}{\ell}$. Call the set of all such imaginary elements $\mathscr{B}$. We easily compute the rank of $\mathscr{B}$ to be 1. Let $\varepsilon_1 \in P \times \mathscr{B}$ be such that $A\, \varepsilon_1 \mathscr{g}$ iff there is a $\ell \in \mathscr{F}$, $a\, \varepsilon_0\, \ell$, and a is in the degree-1 component of $\underset{\sim}{\ell}$ named by $\mathscr{g}$. Let $E(x, y) \in FE(\varnothing)$ be defined on $\mathscr{B}$ so that

$$\mathscr{g}, \mathscr{g}' \in \mathscr{B} \to E(\mathscr{g}, \mathscr{g}') \quad \text{iff} \quad \neg\exists x\big(x\, \varepsilon_1 \mathscr{g} \wedge x\, \varepsilon_1 \mathscr{g}'\big).$$

E partitions $\mathscr{B}$ into m strongly minimal components. Notice that $\operatorname{acl}(a) \cap \mathscr{B}$ consists of m independent elements, one from each component of $\mathscr{B}$. This example shows that when $k = 1$ the rank of P may be any natural number.

Observe that M is $\aleph_0$-categorical, $\aleph_0$-stable, and transitive. Proposition 1.5 implies there is a coordinatization in the sense defined in that section. Following the outlines given after Proposition 1.5 will yield our $\mathscr{B}$ as the rank 1 set of coordinates. So there is some relationship between our notion of coordinatization and the one used in **[CHL]**.

3. Proofs of the theorems. In this section the symbols H, $\mathfrak{F}$, σ, ε, P, n and k denote the same objects as they did in §2. Also, (1)–(7) denote the lines previously enumerated.

LEMMA 3.1. *$\mathfrak{F}$ is the set of realizations of a complete stationary type over $\varnothing$ in M^{eq}, whenever H is stationary.*

PROOF. The elements of $\mathfrak{F}$ are names for the sets $\sigma(M, \bar{a})$, where $\bar{a}$ is an independent sequence of elements of H. Since H is stationary, $t(\bar{a})$ is stationary. The lemma now follows from Lemma 1.1.

This is one good reason for choosing σ to be a canonical representative. Another reason is that since $\sigma(x, \bar{y})$ is satisfied on a set of indiscernibles, there is some invariance under permutation of the variables which will make some of the proofs easier.

Let q_0 denote the type defining $\mathfrak{F}$.

To prove Lemma 2.2 we use Normalization Theorem 1.4. Our basic formula is $a\,\varepsilon\, x$ for $a \in H$. For such a we let P_a be the set R_a defined in (4). Let $\mathscr{P} = \{P_a\colon a \in H\}$.

LEMMA 3.2. *$\mathscr{P}$ satisfies the basic assumptions* (N1)–(N4).

PROOF. (N1), (N2), and (N4) are easy to verify. To prove (N3) it suffices to show

Claim. Every $q \in P_a$ does not fork over a.

Since $q_0 \cup \{a\,\varepsilon\, x\}$ forks over $\varnothing$, every $q \in P_a$ forks over $\varnothing$. Thus, $U(q) \leqslant U(q \upharpoonright a) < \gamma + 1$. But $U(q) = \gamma$, so $U(q \upharpoonright a)$ must also be γ, which proves the Claim.

The Normalization Theorem now implies

LEMMA 3.3. *There is a formula $y\,\varepsilon^*\, x$ such that, for all $a, b \in H$,*

(i) *for all $q \in \bigcup \mathscr{P}$, $q \in P_a$ iff $a\,\varepsilon^*\, x \in q$,*

(ii) $P_a = P_b$ *iff* $\models \forall x(a\,\varepsilon^*\, x \leftrightarrow b\,\varepsilon^*\, x)$.

Without loss of generality we assume σ was chosen so that $\varepsilon = \varepsilon^*$. This proves Lemma 2.2.

We assume from here on that H is stationary.

Let E and n be as defined in (5) and (7), respectively.

LEMMA 3.4. *There is a β such that $\beta + n = \alpha$ and $U(a/(a/E)) = \beta$ for $a \in H$.*

PROOF. As in $\underset{s}{(7)}$ let $\langle \mathscr{b}_0, \ldots, \mathscr{b}_n \rangle = \bar{\mathscr{b}} \in \underset{\sim}{a}$ be independent over $\varnothing$.

Claim 1. If $a \equiv c(\bar{\mathscr{b}})$, then $E(a, c)$.

To prove this claim we use the fact that $E(a, c)$ iff $R_a = R_c$. Let $\mathscr{b} \in \underset{\sim}{a}$ be such that $\mathscr{b} \downarrow_a \bar{\mathscr{b}}c$. It suffices to show that $\mathscr{b} \in \underset{\sim}{c}$; for then a nonforking extension of $t(\mathscr{b}/a)$ over ac contains $c\,\varepsilon\, x$. We know $U(\mathscr{b}/a\bar{\mathscr{b}}c) = \gamma$, where $U(\mathscr{b}/\varnothing) = \gamma + 1$. By the definition of n, $\mathscr{b} \underset{s}{\not\downarrow} \bar{\mathscr{b}}$, which implies $U(\mathscr{b}/\bar{\mathscr{b}}) < \gamma + 1$. Thus, $U(\mathscr{b}/a\bar{\mathscr{b}}c) = U(\mathscr{b}/\bar{\mathscr{b}})$, i.e., $\mathscr{b} \downarrow_{\bar{\mathscr{b}}} ac$. Since $a \equiv c(\bar{\mathscr{b}})$, we have $a \equiv c(\mathscr{b}\bar{\mathscr{b}})$, proving $\mathscr{b} \in \underset{\sim}{c}$ and the claim.

Claim 2. $a/E \in \operatorname{acl}(\bar{\mathscr{b}})$.

Suppose there are infinitely many different classes in $\{c/E\colon c \equiv a(\bar{\mathscr{b}})\}$. That an element d is c/E for some c realizing $t(a/\bar{\mathscr{b}})$ is a property of $t(d/\bar{\mathscr{b}})$. Thus, using compactness we can find arbitrarily many such classes, contradicting Claim 1.

Claim 3. For all $m \leqslant n$ and c with $\bar{\mathscr{b}} \in \underset{\sim}{c}$, $U(c/\mathscr{b}_0, \ldots, \mathscr{b}_m) + m = \alpha$.

We prove Claim 3 by induction on m. It is true for $m = 0$ by Lemma 1.2(i). Suppose it holds for $m = l$. We have $U(\mathscr{b}_{l+1}) = U(\mathscr{b}_{l+1}/c) + 1$. Since $\mathscr{b}_{l+1} \downarrow_c \mathscr{b}_0, \ldots, \mathscr{b}_l$ and $\mathscr{b}_{l+1} \downarrow_{\varnothing} \mathscr{b}_0, \ldots, \mathscr{b}_l$, we have

$$U(\mathscr{b}_{l+1}/\mathscr{b}_0, \ldots, \mathscr{b}_l) = U(\mathscr{b}_{l+1}/c\mathscr{b}_0, \ldots, \mathscr{b}_l) + 1.$$

By Lemma 1.2(i), $U(c/\mathscr{b}_0, \ldots, \mathscr{b}_l) = U(c/\mathscr{b}_0, \ldots, \mathscr{b}_{l+1}) + 1$. By the inductive hypothesis, $U(c/\mathscr{b}_0, \ldots, \mathscr{b}_{l+1}) + l + 1 = \alpha$ which proves the claim.

Let β be such that $\beta + n = \alpha$.

Claim 4. $U(a/(a/E)) = \beta$.

Let $d = a/E$. Suppose $c \equiv a(d)$ and $c \downarrow_d a\bar{\mathscr{b}}$. Then $U(a/d) = U(c/d) = U(c/da\bar{\mathscr{b}})$. Since d is algebraic over a, we have $c \downarrow_a \bar{\mathscr{b}}$. Then $E(a, c)$ implies $\bar{\mathscr{b}} \in \underset{\sim}{c}$.

By Claim 3, $U(c/\bar{\ell}) + \beta$. By Claim 2, $c \downarrow_{\bar{\ell}} d$, so $U(c/d\bar{\ell}) = U(c/\bar{\ell}) = \beta$; thus, $U(c/d) = \beta$, which proves Claim 4 and Lemma 3.4.

This is the reason we use U-rank instead of one of the continuous rank notions—they do not obey the nice identities given in Lemma 1.2.

We now turn our attention to $P = H/E$, ε_0 defined in (6) above, and $\mathfrak{G} = \langle P, \mathfrak{F}, \varepsilon_0 \rangle$. Let $\underset{\sim}{\ell}$ now denote $\{a/E \in P: a \in H \text{ and } a \in \underset{\sim}{\ell}\}$.

LEMMA 3.5. (i) *P is the set of realizations of a complete stationary type with* $U(P) = n + 1$.

(ii) *If $m \leqslant n$, $\ell_0, \ldots, \ell_m$ are independent, and $a \in \bigcap_{i \leqslant m} \underset{\sim}{\ell}_i$, then $U(a/\bar{\ell}) = n - m$.*

(iii) *If $m \leqslant n$, $\langle \ell_0, \ldots, \ell_m \rangle = \bar{\ell} \in \underset{\sim}{a}$, and $\bar{\ell}$ is independent over $\varnothing$, then $U(a/\bar{\ell}) = n - m$.*

PROOF. (i) The proof that P is the set of realizations of a complete stationary type is like the proof of Lemma 3.1. Let $d \in P$ be a/E for $a \in H$. By Lemma 1.2, $U(ad) = U(d/a) + U(a) = \alpha + 1$. We also know that

$$U(a/d) + U(d) \leqslant U(ad) \leqslant U(a/d) \oplus U(d).$$

Since $U(a/d) = \beta$, we conclude quickly that $U(d) = n + 1$.

(ii) To simplify notation suppose $m = 0$. Let $a \in \underset{\sim}{\ell}_0$ be d/E so that $U(d/\ell_0) = \alpha$ and $U(d/a\ell_0) = \beta$. By Lemma 1.2 again, $U(ad/\ell_0) = \alpha$. By an argument similar to that given in (i), we find $U(a/\ell_0) = n$.

(iii) Similar to the proof of (ii).

We now turn our attention to $\mathfrak{G}$ and continue the proof of Theorem 1. First we consider the case $n = 0$. Notice that then $U(P) = 1$. Thus, the theorem is true in this case.

From now on in this section we assume $n > 0$.

Our next immediate goal is Proposition 3.8. It is this result which gives the $\aleph_0$-stability in the Theorem in the Abstract. For motivation suppose $n = 1$. Then if $\ell_0 \downarrow \ell_1$, $\underset{\sim}{\ell}_0 \cap \underset{\sim}{\ell}_1$ is finite and nonempty; $a \in \underset{\sim}{\ell}_0$ implies $U(a/\ell_0) = 1$. We will show that for each $a \in \underset{\sim}{\ell}_0$ there is a $b \in \underset{\sim}{\ell}_0 \cap \underset{\sim}{\ell}_1$ with $a \overset{\mathrm{s}}{\equiv} b$ (ℓ_0). It follows that $x \,\varepsilon\, \ell_0$ has finitely many nonalgebraic completions in P, each of finite multiplicity. In other words it has Morley rank 1 relative to P.

LEMMA 3.6. *Suppose $a \in P$ with $U(a/\bar{d}) = m$, some $m \leqslant n$. Then there are $\ell_0, \ldots, \ell_{m-1}$ with $\langle \ell_0, \ldots, \ell_{m-1} \rangle \in \underset{\sim}{a}$, $\{\ell_0, \ldots, \ell_{m-1}, \bar{d}\}$ independent over $\varnothing$, and $a \in \operatorname{acl}(\bar{\ell}\bar{d})$.*

PROOF. Choose an independent sequence $\ell_0, \ldots, \ell_n$ with $\bar{\ell} \in \underset{\sim}{a}$ and $\bar{\ell} \downarrow_a \bar{d}$. Assume the enumeration is such that $\bar{d}, \ell_0, \ldots, \ell_l$ are independent over $\varnothing$ and, for $l < i \leqslant n$, $\ell_i \not\downarrow d\ell_0, \ldots, \ell_l$. We *claim* $U(a/\bar{\ell}\bar{d}) = m - (l + 1)$. That $U(a/\ell_0, \ldots, \ell_l, \bar{d}) = m - (l + 1)$ is proved exactly like Claim 3. So, it suffices to show that

(i) for $l \leqslant i \leqslant n$,

$$\ell_{i+1} \underset{\bar{d}\ell_0, \ldots, \ell_i}{\not\downarrow} a.$$

Notice that $U(\mathscr{b}_{i+1}/a) = U(\mathscr{b}_{i+1}) - 1$, and, since $\bar{\mathscr{b}} \in \underset{\sim}{a}$ and $\bar{\mathscr{b}} \downarrow_a \bar{d}$, we have

$$U(\mathscr{b}_{i+1}/a\mathscr{b}_0,\ldots,\mathscr{b}_i\bar{d}) = U(\mathscr{b}_{i+1}/\varnothing) - 1.$$

Since $\mathscr{b}_{i+1} \not\downarrow \mathscr{b}_0,\ldots,\mathscr{b}_i\bar{d}$, we have $U(\mathscr{b}_{i+1}/\mathscr{b}_0,\ldots,\mathscr{b}_i\bar{d}) < U(\mathscr{b}_{i+1}/\varnothing)$. Thus,

$$U(\mathscr{b}_{i+1}/a\mathscr{b}_0,\ldots,\mathscr{b}_i\bar{d}) = U(\mathscr{b}_{i+1}/\mathscr{b}_0,\ldots,\mathscr{b}_i\bar{d});$$

i.e.,

$$\mathscr{b}_{i+1} \underset{\bar{d}\mathscr{b}_0,\ldots,\mathscr{b}_i}{\downarrow} a,$$

which proves the claim.

By Lemma 3.5 (iii) a is algebraic over $\bar{\mathscr{b}}$. Thus, $l = m - 1$. By the claim the lemma is proved.

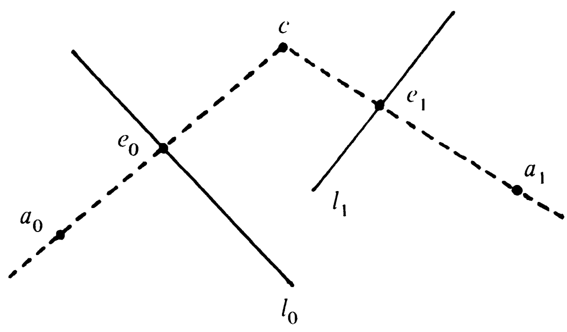
LEMMA 3.7. *Suppose $a \in P$ with $U(a/\bar{d}) = m$ for some $m \leqslant n$, and $\{\mathscr{b}_0,\ldots,\mathscr{b}_{m-1}, \bar{d}\}$ is independent over $\varnothing$. Then there is a b with $\langle \mathscr{b}_0,\ldots,\mathscr{b}_{m-1}\rangle \in \underset{\sim}{b}$ and $b \equiv a\ (\bar{d})$.*

PROOF. The lemma follows from Lemma 3.6, using an automorphism. Let $\langle \mathscr{g}_0,\ldots,\mathscr{g}_{m-1}\rangle \in \underset{\sim}{a}$ be such that $\{\mathscr{g}_0,\ldots,\mathscr{g}_{m-1}, \bar{d}\}$ is independent. Since the type over $\varnothing$ of the blocks is stationary, $\bar{\mathscr{b}} \equiv \bar{\mathscr{g}}(\operatorname{acl}(\bar{d}))$. Let b be the image of a under an automorphism which fixes $\operatorname{acl}(\bar{d})$ and sends $\bar{\mathscr{g}}$ to $\bar{\mathscr{b}}$. This proves the lemma.

PROPOSITION 3.8. *Suppose $\mathscr{b}_0,\ldots,\mathscr{b}_m$ are independent. Then the formula $\bigwedge_{i\leqslant m} x\ \varepsilon\ \mathscr{b}_i$ has finitely many completions in P of rank $n - m$, each of finite multiplicity. Also, this formula has no completion of rank $> n - m$.*

PROOF. Here $\bar{\mathscr{b}}$ denotes the sequence $\mathscr{b}_0,\ldots,\mathscr{b}_m$. That the highest possible rank over $\bar{\mathscr{b}}$ for an element satisfying $\bigwedge_{i\leqslant m} x\ \varepsilon\ \mathscr{b}_i$ is $n - m$ is computed as usual. Find $\mathscr{b}_{m+1},\ldots,\mathscr{b}_n$ with $\{\mathscr{b}_0,\ldots,\mathscr{b}_n\}$ independent. If $a \in P$ satisfies $\bigwedge_{i\leqslant m} x\ \varepsilon\ \mathscr{b}_i$ and $U(a/\bar{\mathscr{b}}) = n - m$, then $a \in \bigcap_{i\leqslant m} \underset{\sim}{\mathscr{b}_i}$. By Lemma 3.7 there is a $b \in \bigcap_{i\leqslant n} \underset{\sim}{\mathscr{b}_i}$ with $a \equiv b\ (\bar{\mathscr{b}})$. The proposition follows from the fact that $\bigcap_{i\leqslant n} \underset{\sim}{\mathscr{b}_i}$ is finite.

The next lemma is the main step in showing that $P = \operatorname{acl}(A)$ for some A of rank 1. The motivation is the following. Suppose $(q, L; I)$ is a projective plane. For $l \in L$ let $[l]$ denote $\{a \in Q\colon aIl\}$. Then for $l_0 \neq l_1 \in L$; $a_0 \neq a_1 \in Q$ with $a_i \notin [l_j]$ for $i, j = 0, 1$, we have $Q = \operatorname{acl}([l_0] \cup [l_1] \cup a_0 \cup a_1)$. This is seen by examining the following picture:



Claim. For $c \in P$ and $i \leqslant n$ there is an independent sequence a_i of realizations of P, and $e_i \in P$ such that

(i) $e_i \cup a_i$ and $c \cup a_i$ are independent,

(ii) $\{e_i a_i : i \leqslant n\}$ is independent,

(iii) $U(c/e_i a_i) = n$ and $c \in \operatorname{acl}(e_0 a_0 \cup \cdots \cup e_n a_n)$.

PROOF. Choose independent $\ell_0, \ldots, \ell_n$ and $c \in \bigcap_{i \leqslant n} \underset{\sim}{\ell}_i$ as in (7). As in §1.3 there is a Morley sequence I_i in $\operatorname{stp}(c/\ell_i)$ such that

$$c \underset{I_i}{\downarrow} \ell_i \qquad \text{and} \qquad c \underset{\ell_i}{\downarrow} I_i .$$

Choosing a minimal such Morley sequence yields the independence of I_i since $U(c/\ell_i) = n$. We may require that

$$I_i \underset{\ell_i}{\downarrow} \bigcup \{ I_j \cup \ell_j : j \neq i \}.$$

The independence of $\{\ell_i : i \leqslant n\}$ then yields the independence of $\{I_i : i \leqslant n\}$. A U-rank calculation shows that $c \in \operatorname{acl}(\bigcup_i I_i)$. Letting e_i be any element of I_i and $a_i = I_i \setminus \{e_i\}$ proves the claim.

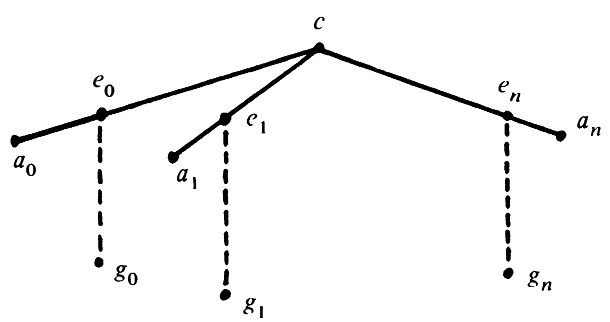
Notation. Let $l_i = |a_i|$.

LEMMA 3.9. *Suppose $c \in P$ and, for $i \leqslant n$, $\mathscr{g}_i$ is an independent sequence of n elements of $\mathfrak{F}$, and a_i is an independent sequence of elements of P of length l_i. Furthermore, suppose that $c \cup \{\mathscr{g}_0, \ldots, \mathscr{g}_n\} \cup \{a_0, \ldots, a_n\}$ is independent. Then for $i \leqslant n$ there is $e_i \in P$ such that*

(i) *$t(e_i/\mathscr{g}_i)$ has U-rank 1 and finite multiplicity,*

(ii) *c is algebraic over $\bar{e}\bar{a}$ in $\mathfrak{S}$.*

PROOF. The picture we are trying to produce is



We will do the proof backwards. First we pick the a_i's and e_i's so that $c \in \operatorname{acl}(\bar{a}\bar{e})$. Then we choose the $\mathscr{g}_i$'s and show that $c\bar{\mathscr{g}}\bar{a}$ is independent. The lemma follows since $t(c/\varnothing)$, $t(\mathscr{g}_i/\varnothing)$, and $t(a_i/\varnothing)$ are all stationary, and the desired property is a property of the types.

Let c and $\{e_i a_i : i \leqslant n\}$ be as in the claim. By Lemma 3.6 for $i \leqslant n$ there is $\mathscr{g}_i \in \underset{\sim}{e}_i$, an independent sequence of n elements of $\mathfrak{F}$ with $\mathscr{g}_i \downarrow_{\varnothing} c a_i$, and $e_i \in \operatorname{acl}(a_i c \mathscr{g}_i)$. By

Proposition 3.8, $U(e_i/\mathscr{g}_i) = 1$ and $t(e_i/\mathscr{g}_i)$ has finitely many strong types. Furthermore, we may choose the $\mathscr{g}_i$'s so that

(A) $$e_i\mathscr{g}_i \underset{\varnothing}{\downarrow} c\bar{e}\bar{\mathscr{g}}\bar{a}\setminus\{e_i\mathscr{g}_ia_i\},$$

using the fact that $e_i \downarrow_{\varnothing} c\bar{a}\bar{e}\setminus\{e_ia_i\}$.

We have $e_i\mathscr{g}_ia_i$'s with the desired properties (i) and (ii). It suffices to show that $c\bar{\mathscr{g}}\bar{a}$ is independent. Assume, towards a contradiction, that this sequence is dependent. Since $a_i \downarrow_{\varnothing} c\bar{\mathscr{g}}$ we know that for some $i \leqslant n$,

$$a_i \underset{c\bar{\mathscr{g}}}{\not\downarrow} \bar{a}\setminus\{a_i\}.$$

Say $i = n$. By the way all these elements were chosen it is easy to check that

$$a_n \underset{ce_n}{\downarrow} \bar{\mathscr{g}}\bar{a}\bar{e}\setminus\{a_ne_n\}.$$

Thus,

$$U(a_n/c\bar{e}\bar{\mathscr{g}}a_0 \cdots a_{n-1}) = U(a_n/ce_n) = U(a_n) - 1.$$

We conclude that $U(a_n/c\bar{e}\bar{\mathscr{g}}a_0 \cdots a_{n-1}) = U(a_n/c\bar{\mathscr{g}}a_0 \cdots a_{n-1})$, implying

(B) $$a_n \underset{c\bar{\mathscr{g}}a_0 \cdots a_{n-1}}{\downarrow} e_n.$$

We choose the $\mathscr{g}_i$'s so that $e_n \in \operatorname{acl}(ca_n\mathscr{g}_n)$, so (B) implies $e_n \in \operatorname{cl}(c\bar{\mathscr{g}}a_0 \cdots a_{n-1})$. Thus,

$$e_n \underset{\mathscr{g}}{\not\downarrow} c\bar{\mathscr{g}}\bar{a}\setminus\{\mathscr{g}_na_n\}.$$

We conclude that

$$e_n\mathscr{g}_n \underset{\varnothing}{\not\downarrow} c\bar{\mathscr{g}}\bar{a}\setminus\{\mathscr{g}_na_n\},$$

in contradiction to (A). This proves the lemma.

For $i \leqslant n$ let $r^i = t(e_i/\mathscr{g}_i)$. The above lemma indicates that

$$\operatorname{acl}\big(r^0(P) \cup \cdots \cup r^n(P) \cup a_0 \cup \cdots \cup a_n\big)$$

is almost the whole model. The only points not in this set are those b such that $b \downarrow_{\varnothing} \bar{\mathscr{g}}\bar{a}$. To handle this pathology we use the notion of weight (see [**M**, D.1, or **Sh**, V, 3.12]). Let w be the weight of P and $\{\bar{\mathscr{g}}^0\bar{a}^0,\ldots,\bar{\mathscr{g}}^w\bar{a}^w\}$ an independent set of realizations of $t(\bar{\mathscr{g}}\bar{a})$.

Let r_{ij} denote the image of r over $\mathscr{g}^i_j$, while $\bar{r}_i(P)$ denotes $r_{i0}(P) \cup \cdots \cup r_{in}(P)$.

LEMMA 3.10. $P = \operatorname{acl}(\bar{r}_0(P) \cup \cdots \cup \bar{r}_w(P) \cup \bar{a}^0 \cup \cdots \cup \bar{a}^w)$.

PROOF. Given $b \in P$, the definition of weight implies there is $i \leqslant w$ such that $b \downarrow_{\varnothing} \bar{\mathscr{g}}^i\bar{a}^i$. By Lemma 3.9, $b \in \operatorname{acl}(\bar{\mathscr{g}}^i\bar{a}^i)$.

This completes the proof of Theorem 1.

It is worth summarizing what was proven in the last couple of lemmas, especially since we are dealing with formulas in two different structures, M^{eq} and $\mathfrak{G}$. Let L_2 be the language of $\mathfrak{G}$.

COROLLARY 3.11. *There is $\rho(x, y)$, an open formula in L_2 such that the following hold*:

(i) *$\rho(P, \mathscr{g})$ has finitely many nonalgebraic completions relative to P in M, each of U-rank* 1 *and having finite multiplicity.*

(ii) *There is a formula $\theta(x, \bar{y}\bar{z}) \in L_2$ such that for all $b \in P$ there is an $i \leqslant w$ and $e_j \in \rho(P, \mathscr{g}^i_j)$ for $j \leqslant n$ such that $\mathfrak{S} \models \exists! x\, \theta(x, \bar{e}\bar{\mathscr{g}}^i)$ and b satisfies $\theta(x, \bar{e}\bar{\mathscr{g}}^i)$.*

In the case when H is definable we get the following corollary to Theorem 1.

THEOREM 2. *Suppose the H from* (1) *is definable. Then P is also definable and is in the algebraic closure of a set of Morley rank* 1.

THEOREM 3. *Suppose M is superstable, transitive, and stationary with $U(M) = \alpha + 1$ for some α. Then one of the following holds*:

(i) *There is a* 0*-definable equivalence relation $E \subset M^2$ with infinite classes.*

(ii) *M is the algebraic closure of a set of Morley rank* 1.

PROOF. If $\alpha = 0$ then (ii) certainly holds. Otherwise we carry out the coordinatization process with $H = M$. If (i) does not hold, the equivalence relation E defined in (5) must have finite classes. That is, every $a \in M$ is in $\mathrm{acl}(a/E)$. $P = \{a/E: a \in M\}$ is the algebraic closure of a set of Morley rank 1 by Theorem 2. This proves Theorem 3.

There is still more to be said about the case $k = 1$, $n > 0$. Recall that this is always the case when M is $\aleph_0$-categorical, $\aleph_0$-stable [**CHL**, 6.3]. Then we obtain the Coordinatization Theorem (Proposition 1.5). We now show that whenever $k = 1$ we can obtain a similar theorem for P. The only essential difference between our method of coordinatizing and the one yielding Proposition 1.5 is that in a superstable theory we may not be able to pick a formula of Morley degree 1 in H. However, in the quotient P we can come close to this.

PROPOSITION 3.12. *Suppose $k = 1$. Then there is $\mathfrak{F}' \subset M^{\mathrm{eq}}$ of U-rank* 1 *such that $a \in H$ implies $\mathrm{acl}(a) \cap \mathfrak{F}' \neq \varnothing$.*

We assume for the remainder of the paper that $k = 1$ and $n > 0$. As a special case of Proposition 3.8 we have

LEMMA 3.13. *The formula $x \,\varepsilon\, \ell$ has finitely many extensions in P of U-rank n, each of finite multiplicity.*

By Lemma 3.13 there is $\psi(x, \mathscr{k})$, a formula over $\mathrm{acl}(\ell)$ whose only extension over M of U-rank n is $t((a/E)/M)$ for $t(a/M) \cong p$ from (1). Using Normalization Theorem 1.4 we may assume that whenever $\mathscr{k}' \equiv \mathscr{k}$ and $\psi(x, \mathscr{k}) \wedge \psi(x, \mathscr{k}')$ has an extension of U-rank n, $\mathscr{k} = \mathscr{k}'$. The proposition follows directly from

LEMMA 3.14. *There is $b \in P$ with $\mathscr{k} \in \mathrm{acl}(b)$. Also $U(\mathscr{k}) = 1$.*

PROOF. Suppose $b \in \psi(P, \mathscr{k})$ with $U(b/\mathscr{k}) = n$. Let $q \in S(M)$ be the extension of $\psi(x, \mathscr{k})$ of U-rank n consistent with P. By the definition of complexity there is a canonical representative of q over b. In fact, there is $\mathscr{k}' \in \operatorname{acl}(b)$ such that $\psi(x, \mathscr{k}') \in q$ and $\mathscr{k}' \equiv \mathscr{k}$. It follows that $\mathscr{k}' = \mathscr{k}$, so $\mathscr{k} \in \operatorname{acl}(b)$. This implies $U(\mathscr{k}b) = U(b) = n + 1$. The lemma follows since $U(\mathscr{k}b)$ is also $U(b/\mathscr{k}) + U(\mathscr{k})$.

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